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Robot Kinematics Using Dual Quaternions

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ABSTRACT

From the point of view of classical mechanics, deriving the equations of motion for systems of coupled rigid bodies is regarded as a straightforward procedure: once a suitable set of generalized coordinates and reference frames have been chosen, what remains is to either apply Lagrange's equations or Newton and Euler's equations to obtain the differential equations of motion. As the complexity of multibody system increases, the need for more elegant formulation of the equation of motion becomes an issue of paramount importance. Our primary focus is on the kinematic analysis of rigid bodies and serial manipulators (robotic systems) using simultaneously, both homogeneous transformations (4x4) matrices and Dual Quaternions, for the sake of results comparisons (cost, complexity, storage capacity etc.). This paper has been done mainly for educational and pedagogical purposes, hoping that the scientific community will finally adopt and use Dual Quaternions at least when dealing with multibody systems and specially robotics.

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Symbols :

p,q, r : Quaternions, $\hat{q}$, $\hat{S}$,: Dual quaternions (DQ), q^{-1}: Inverse of a quaternion, $\hat{q}^{-1}$: Inverse of a DQ
q^*: Quaternion conjugate, $\bar{\hat{q}}^*$::Dual Quaternion conjugate , $\otimes$:Multiplication operator ,
S :sin($\frac{\alpha}{2}$), C : cos($\frac{\alpha}{2}$) , $(\alpha), (\theta)$, (β) : Angles, Amplitudes, $\vec{\boldsymbol{T}}$, $\vec{\boldsymbol{t}}$, $\vec{\boldsymbol{d}}$: Translations vectors,
$\boldsymbol{v}$, $\overrightarrow{\boldsymbol{v}'}$, $\vec{n}$, **n,** $\vec{l}$ **:** Vectors, [A] ,[R], [I], [T] : Matrices, **i , j , k** : Unitary vectors,
ε, I : the dual factors, EE : End effector, W :Wrist

1. INTRODUCTION

A robot is usually designed so that it can position its end-effector with a three degree-of-freedom of translation and three degree-of-freedom of orientation within its workspace. Kinematic analysis studies the relative motions, such as the displacement, velocity, and acceleration, associated with the links of a given robot. Although screw theory based solution methods have been widely used in many robotic applications , the elements of screw theory can be traced to the work of Chasles and Poinsot in the early 1800s. Using the theorems of Chasles and Poinsot as a starting point, Robert S. Ball developed a complete theory of screws which he published in 1900 [18].

Throughout the development of kinematics, numerous mathematic theories [17] and tools have been introduced and applied. Examples are the (4×4) homogeneous transformation matrices introduced by Denavit and Hartenberg (1955) [1],Lie groups and Lie Algebra (Selig, J.M [13], [15]) and quaternions and dual

quaternions introduced by Yang and Freudenstein (1964) [2], also see Bottema and Roth (1979) [3] and McCarthy (1990) [4].
This paper is organised as follows : Chapter II defines the concept of quaternions and dual quaternions (DQ), in chapter III using this concept and simultaneously (4x4) homogeneous matrices ,basic elementary transformations examples are treated while some more theoretical background concerning their corresponding centers of rotation and moments is given in chapter IV.

Kinematic equations :A) Denavit-Hartenberg convention and B) Jacobian matrix in dual quaternion space are given in chapter V. We are concerned in chapter VI with the kinematics of a planar RRR manipulator, a3R wrist and a 6R Puma type robot and finally in the last chapter some important conclusions are drawn.

2. QUATERNIONS AND DUAL QUATERNIONS (DQ)

1) Quaternions or rotation representation

Quaternions were first discovered and described by the Irish mathematician Sir Rowan Hamilton in 1843. Indeed quaternion's representation and axis-angle representation are very similar.

Both are represented by the four dimensional vectors. Quaternions also implicitly represent the rotation of a rigid body about an axis. It also provides better means of key frame interpolation and doesn't suffer from singularity problems [7],[16].

The definition of a quaternion can be given as $(s, \boldsymbol{m})$ or (s, x, y, z) where **m** is a 3D vector, so quaternions are like imaginary (complex) numbers with the real scalar part s and the imaginary vector part **m**. Thus it can be also written as: $s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
There are conversion methods between quaternions, axis-angle and rotation matrix.
Common operations such as addition, inner product etc can be defined over quaternions.
Given the definition of q_1 and q_2:
$q_1 = s_1 + x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ or $q_1 = (s_1, \boldsymbol{m_1})$
$q_2 = s_2 + x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ or $q_2 = (s_2, \boldsymbol{m_2})$
Addition operation is defined as:

$$q_1 + q_2 = (s_1 + s_2,\ \boldsymbol{m_1} + \boldsymbol{m_2}) = (s_1 + s_2) + (x_1 + x_2)\boldsymbol{i} + (y_1 + y_2)\boldsymbol{j} + (z_1 + z_2)\boldsymbol{k}$$

dot (scalar, inner): product operation(.) as:
$q_1.q_2 = s_1.s_2 + \boldsymbol{m_1}.\boldsymbol{m_2}$
Quaternion multiplication is non commutative, but it is associative. Multiplication identity element is defined as $(1, (0,0,0))$
We can also perform the multiplication in the imaginary number domain using the definitions:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\boldsymbol{1}$$
$$\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{k} \ , \qquad \boldsymbol{j}.\boldsymbol{k} = \boldsymbol{i} \ , \qquad \boldsymbol{k}.\boldsymbol{i} = \boldsymbol{j}$$
$$\boldsymbol{j}.\boldsymbol{i} = -\boldsymbol{k} \ , \qquad \boldsymbol{k}.\boldsymbol{j} = -\boldsymbol{i} \ , \qquad \boldsymbol{i}.\boldsymbol{k} = -\boldsymbol{j}$$

Equations (**1**) to (**12**) state the definitions, rules and properties of dual quaternion algebra.
Quaternion multiplication (⊗)is defined as:

$$\boxed{q_1 \otimes q_2 = (s_1.s_2 - \boldsymbol{m_1}.\boldsymbol{m_2}, s_1.\boldsymbol{m_2} + s_2.\boldsymbol{m_1} + \boldsymbol{m_1} \wedge \boldsymbol{m_2})} \qquad (1)$$

Each quaternion has a conjugate q^* and an inverse (except zero quaternion) defined by:

$$q^* = (s, -\boldsymbol{m}) \qquad (2)$$

and an inverse $q^{-1} = \left(\frac{1}{|q|}\right)^2 q^*;\ (q \neq 0)$
Where $\boxed{|q|^2 = s^2 + \boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 = q \otimes q^* = q^* \otimes q}$

Rotations are defined by unit quaternions.Unit quaternions must satisfy $|q| = 1$
Since multiplication of two unit quaternions will be a unit quaternion, N rotations can be combined into one unit quaternion $c_R = q_{R1}.q_{R2}.q_{R3} \ldots . q_{RN}$
It is also possible to rotate a vector directly by using quaternion multiplication. To do this, we must define 3D vector $\boldsymbol{V} = (v_x, v_y, v_z)$ that we want to rotate in quaternion definition as $q_v = (0, \boldsymbol{v}) = 0 + v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$ .The rotated vector $\boldsymbol{V'} = (v_x', v_y', v_z')$ can be defined as $q_{v'} = (0, \boldsymbol{v'}) = 0 + v_x'\boldsymbol{i} + v_y'\boldsymbol{j} + v_z'\boldsymbol{k}$

Noting that, in quaternion rotation equation $q_R^{-1} = q_R^*$ (For unit quaternion).So, rotation of q_v by quaternion q_R can be calculated as: $\boxed{q_v' = q_R \otimes q_v \otimes q_R^{-1} = q_R \otimes q_v \otimes q_R^*}$ (3)

And two rotations can be applied to the vector **V** such as:

$$\boxed{q_v' = p_R \otimes (q_R \otimes q_v \otimes q_R^{-1}) \otimes p_R^{-1}} \quad (4)$$

$c_R = p_R \otimes q_R$ which means: $q_v' = c_R \otimes q_v \otimes c_R^{-1}$

Where q_R and p_R are rotation quaternions and c_R is the combined rotation quaternion.The equation implies that vector $\boldsymbol{V}$ is first rotated by the rotation represented by q_R and then p_R.
Thus, quaternion q_R that defines a rotation about (around) the axis **a** denoted by the unit vector (a_x, a_y, a_z)) of an angle α could be written as :

$$\boxed{q_R = cos\ \frac{\alpha}{2} + sin\ \frac{\alpha}{2}\ (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k})} \quad (5)$$

$q_R = -q_R = rot\ (\alpha, \vec{\boldsymbol{a}}) = rot\ (-\alpha, -\vec{\boldsymbol{a}})$

2) Dual quaternions : To represent rigid transformations

Dual Quaternions (DQ) were proposed by William Kingdom Clifford in 1873 [6] .They are an extension of quaternions. They represent both rotations and translations whose composition is defined as a rigid transformation. They are represented by the following eight dimensional vector :

$$\boxed{\hat{q} = (\hat{s}, \widehat{\boldsymbol{m}}) = (s, x, y, z, s_\varepsilon, x_\varepsilon, y_\varepsilon, z_\varepsilon) = (\hat{s}, \hat{x}, \hat{y}, \hat{z})} \quad (6)$$

$\hat{q} = q + \varepsilon q_\varepsilon = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + \varepsilon(s_\varepsilon + x_\varepsilon\boldsymbol{i} + y_\varepsilon\boldsymbol{j} + z_\varepsilon\boldsymbol{k})$

Dual quaternion multiplication is defined by :

$\hat{q}_1 \otimes \hat{q}_2 = q_1 \otimes q_2 + \varepsilon\left(q_1 \otimes q_{2_\varepsilon} + q_{1_\varepsilon} \otimes q_2\right)$; with $\varepsilon^2 = 0$ (7)

ε being the dual factor.

The dual conjugate (analogous to complex conjugate) is denoted by:

$$\boxed{\bar{\hat{q}} = q - \varepsilon q_\varepsilon} \quad (8)$$

This conjugate operator can lead to the definition of the inverse of $\bar{\hat{q}}$ which is :

$\hat{q}^{-1} = \frac{1}{\hat{q}} = \frac{\bar{\hat{q}}}{\bar{\hat{q}}\,\hat{q}} = \frac{1}{q} - \varepsilon\frac{q_\varepsilon}{q^2}$, (a dual number (q = 0) has no inverse)

$$[\hat{q} \otimes \hat{q}^{-1}] = (q + \varepsilon q_\varepsilon)\left(\frac{1}{q} - \varepsilon\frac{q_\varepsilon}{q^2}\right) = \frac{q}{q} - \varepsilon\frac{q_\varepsilon}{q} + \varepsilon\frac{q_\varepsilon}{q} = 1$$

A second conjugation operator is defined for DQs. It is the classical quaternion conjugation and is denoted by: $\hat{q}^* = q^* + \varepsilon q_\varepsilon^*$ where conjugation of dual and non-dual quaternion parts satisfies eq (**2**).
Combining these two conjugation operators will lead to the formalization of DQ transformation on 3D points. Use of both conjugations on $\hat{q}$ can be denoted $\bar{\hat{q}}^*$.
Using definitions (2), (6) and (8) we finally have:

$$\boxed{\bar{\hat{q}}^* = (s, -x, -y, -z, -s_\varepsilon, x_\varepsilon, y_\varepsilon, z_\varepsilon)} \quad (9)$$

DQs naturally inherit some properties of regular quaternions, like:

$\hat{q} = -\hat{q}$ (The same transformation representation)

It is well know that we can use dual quaternions to represent a general transformation subject to the following constraints :

The DQ screw motion operator $\widehat{S} = (q, q_0)$ must be of unit magnitude:

$|\hat{q}| = |\hat{S}| = (q_1 + \varepsilon q_{01})^2 + (q_2 + \varepsilon q_{02})^2 + (q_3 + \varepsilon q_{03})^2 + (q_4 + \varepsilon q_{04})^2 = 1$

This requirement means two distinct conditions or constraints:

$$\begin{cases} q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \\ q_1q_{01} + q_2q_{02} + q_3q_{03} + q_4q_{04} = 0 \end{cases}$$

Which imposed on the eight (8) parameters of a general DQ, effectively reduce the number of degree of freedom $(8 - 2) = 6$; equivalent to the degree of freedom of any free rigid body in 3-D space.(See Appendix II: DQ multiplication)

2-1) <u>Rotations</u>

DQ transformation formula is defined as a series of DQ multiplications which is similar to quaternion transformation; eq (**3**):

$$\boxed{\hat{q}_v{}' = \hat{q}_R \otimes \hat{q}_v \otimes \bar{\hat{q}}^*{}_R = (\hat{q}_R \otimes \hat{q}_v) \otimes \bar{\hat{q}}^*{}_R = \hat{q}_R \otimes (\hat{q}_v \otimes \bar{\hat{q}}^*{}_R)\ ; associativity} \quad (10)$$

For rotation: DQ represents only a 3D rotation ($q_\varepsilon = 0,\ \hat{q}_R = \ \hat{q} = q = R$), it can be used to define a rotation of a 3D vector; $\vec{\boldsymbol{v}} = (v_x, v_y, v_z)$, $\vec{\boldsymbol{v}}$ can also be defined as a DQ of the form:

$\hat{q}_v = 1 + \varepsilon(v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k})$.

Rotation of $\vec{\boldsymbol{v}}$ by $\hat{q}_R$ can be denoted by eq (**10**) where $\hat{q}_v{}'$ represents the DQ form of the transformed vector $\vec{\boldsymbol{v}}' = (v_x{}', v_y{}', v_z{}')$ and can be also represented as:

$\hat{q}_v{}' = 1 + \varepsilon(v_x{}'\boldsymbol{i} + v_y{}'\boldsymbol{j} + v_z{}'\boldsymbol{k})$.

Since $q_\varepsilon = 0$, by def (**9**) and(**2**):

$\bar{\hat{q}}^* = q^*$ and

$\hat{q} \otimes \hat{q}_v \otimes \bar{\hat{q}}^* = q \otimes \hat{q}_v \otimes q^*$ which can be expanded to:

$$\begin{aligned}\hat{q}_v' &= q \otimes \left(1 + \varepsilon(v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k})\right) \otimes q^* \\ &= q \otimes q^* + \varepsilon q \otimes (v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}) \otimes q^* \\ &= 1 + \varepsilon q \otimes (v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}) \otimes q^* \\ &= 1 + \ \varepsilon(v_x{}'\boldsymbol{i} + v_y{}'\boldsymbol{j} + v_z{}'\boldsymbol{k}).\end{aligned}$$

Finally, comparing these results, we finally have:

$$(v_x{}'\boldsymbol{i} + v_y{}'\boldsymbol{j} + v_z{}'\boldsymbol{k}) = q \otimes (v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}) \otimes q^*$$

2-2) **<u>Translations</u>**

DQs also represent translation; a DQ defined as: $\hat{q}_T = 1 + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})$ corresponds to the translation vector $\vec{\boldsymbol{T}} = (t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})$ which could symbolically be noted T . $So\ \ \hat{q}_T = 1 + \varepsilon \frac{T}{2}$

The translation T on the vector $\vec{\boldsymbol{v}}$ can be computed by: $\hat{q}_v' = \hat{q}_T \otimes \hat{q}_v \otimes \bar{\hat{q}}_T^*$

So fortunately using def (**9**) we have $\bar{\hat{q}}_T^* = \hat{q}_T = 1 + \varepsilon \frac{T}{2}$

Then

$$\begin{aligned}\hat{q}_v' = \hat{q}_T \otimes \hat{q}_v \otimes \hat{q}_T &= \left[1 + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})\right] \otimes [1 + \varepsilon(v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k})] \otimes \left[1 + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})\right] \\ &= 1 + \varepsilon[(v_x + t_x)\boldsymbol{i} + (v_y + t_y)\boldsymbol{j} + (v_z + t_z)\boldsymbol{k}]\end{aligned}$$

Which correspond to the transformed vector:

$$\vec{\boldsymbol{v}}' = (v_x + t_x)\boldsymbol{i} + (v_y + t_y)\boldsymbol{j} + (v_z + t_z)\boldsymbol{k}$$

2-3) **<u>Combining rotation and translation</u>**

Transformations represented by DQs can be combined into <u>one</u> DQ (similar to quaternions combination(**4**)). Assuming $\hat{p}$ and $\hat{q}$ are two transformation DQs and q_v is the position vector DQ.

Their combined transformation c can be applied to $\hat{q}_v$:

$$\boxed{\hat{q}_v' = \hat{p} \otimes (\hat{q} \otimes \hat{q}_v \otimes \bar{\hat{q}}^*) \otimes \bar{\hat{p}}^* = (\hat{p} \otimes \hat{q}) \otimes (\hat{q}_v) \otimes (\bar{\hat{q}}^* \otimes \bar{\hat{p}}^*)} \quad (11)$$

$\hat{c} = \hat{p} \otimes \hat{q} \Longrightarrow \hat{q}_v' = \hat{c} \otimes q_v \otimes \bar{\hat{c}}^*$

It is very important (vital) to notice that the most inner transformation of the equation is applied first with an inside to outside manner. In eq (11), $\hat{q}$ is the first transformation followed by the second one $\hat{p}$.

The composition of unit rotation DQ: $\hat{q}_R = q_R$, and a unit translation DQ:

$\hat{q}_T = 1 + \varepsilon(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})$ will give:

$$\boxed{\hat{q}_T \otimes \hat{q}_R = \left(1 + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k})\right) \otimes q_R = q_R + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k}) \otimes q_R} \quad (12)$$

So any unit DQ; $\hat{q} = q + \varepsilon q_\varepsilon$ can be separated to its translation and rotation components: $\hat{q} = q + \varepsilon q_\varepsilon = q_R + \frac{\varepsilon}{2}(t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k}) \otimes q_R = \boxed{R + \varepsilon \frac{TR}{2}}$ (**12** A)

Its inverse is $(R + \varepsilon \frac{TR}{2})^{-1} = R^* - \varepsilon \frac{R^* T}{2}$

Rotation is represented by the non dual part $= q_R = R$, and with a little quaternion math it can be proved that translation $\hat{q}_T$ can be represented by $2q_\varepsilon . q^*$

$$\varepsilon q_\varepsilon = \frac{\varepsilon}{2}\left(t_x\boldsymbol{i} + t_y\boldsymbol{j} + t_z\boldsymbol{k}\right)\otimes q_R \Rightarrow 2q_\varepsilon \otimes q_R^* = \left(t_x\boldsymbol{i} + t_y\boldsymbol{j} + t_z\boldsymbol{k}\right)$$

$$\Rightarrow 2q_\varepsilon \otimes q^* = \left(t_x\boldsymbol{i} + t_y\boldsymbol{j} + t_z\boldsymbol{k}\right)$$

If the translation is applied first: $\boxed{\hat{q} = R + \varepsilon\frac{RT}{2}}$ (**12** B)

Its inverse being $\left(R + \varepsilon\frac{RT}{2}\right)^{-1} = R^* - \varepsilon\frac{TR^*}{2}$

NOTE : From now on, and for more concise and compact writing, we will be using the forms (**12** A) or (**12** B) implicitly using the operator (⊗).

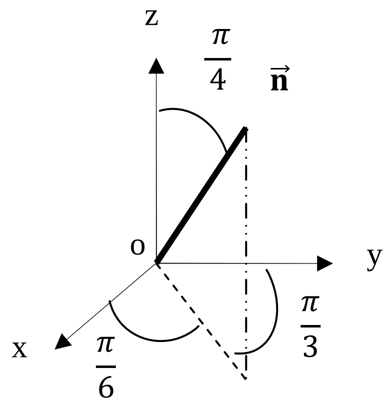
3. <u>ELEMENTARY EXAMPLES</u>

A) Pure rotations:

q = R = ($\cos\frac{\theta}{2}$, $\frac{\sqrt{6}}{4}\sin\frac{\theta}{2}$, $\frac{\sqrt{2}}{4}\sin\frac{\theta}{2}$, $\frac{\sqrt{2}}{2}\sin\frac{\theta}{2}$) = q_r

Observing that: $n_x^2 + n_y^2 + n_z^2 = \frac{6}{16} + \frac{2}{16} + \frac{2}{4} = 1$ we can confirm that this transformation **(Fig. 1)** is a genuine rotation about the 3-D axis **n.** Its counterpart in matrices language is:

$$[\,R\,] = \begin{pmatrix} \frac{3}{8} + \frac{5cos\theta}{8} & \frac{\sqrt{3}(1-cos\theta)}{8} - \frac{\sqrt{2}sin\theta}{2} & \frac{\sqrt{3}(1-cos\theta)}{4} + \frac{\sqrt{2}sin\theta}{4} \\ \frac{\sqrt{3}(1-cos\theta)}{8} + \frac{\sqrt{2}sin\theta}{2} & \frac{1}{8} + \frac{7cos\theta}{8} & \frac{(1-cos\theta)}{4} - \frac{\sqrt{6}sin\theta}{4} \\ \frac{\sqrt{3}(1-cos\theta)}{4} - \frac{\sqrt{2}sin\theta}{4} & \frac{(1-cos\theta)}{4} + \frac{\sqrt{6}sin\theta}{4} & \frac{1}{2} + \frac{cos\theta}{2} \end{pmatrix}$$



(Fig. 1)

Writing this motion in a matrix form will take us a few pages of calculus (manyTRT^{-1} conjugations techniques needed) and a few hours of labour bewaring the eventual errors. We could also use axis -angle method, Rodrigues formula or the conversion technique (from quaternion to matrix)_See (Appendix I),for DQs into matrices /or matrices into DQs. .Matlab programs for computing such conversions have been made. We can thus easily check our matrix [R]. See also Appendix V(its 3x3 upper left matrix)

Comparing the two formats of this transformation:

*Quaternion formulation is more compact; only four terms would describe the motion clearly (Amplitude θ and orientation :rotation axis$\vec{n}$) using general geometry projection techniques.

*Three lines and three columns or nine ‘confusing’ terms compose the 3 by 3 matrix and must satisfy: det(R) = 1

*Quaternions offer a more intuitive frame for using and understanding the effects of three-dimensional rotations because the axis and angle of rotation appear explicitly in their definition.

The rotation axis $\vec{\mathbf{n}}$ could be shifted from the origin O to any other desired position using the applications (see below) : 1) , 2) or 3)

B) Planar movements (displacements):

1) A rotation R followed by a translation $\vec{\boldsymbol{t}}$ applied to the vector $\hat{q}_v = [1 + \varepsilon(0, l, 0)]$ **:**

$\hat{q}_v{}' = \left(R + \frac{\varepsilon}{2}tR\right)\hat{q}_v\overline{\left(R + \frac{\varepsilon}{2}tR\right)}^*$

Ex: rotate $\vec{\boldsymbol{v}}$ an angle Φ about the x-axis, and then translate (displace) it $\vec{\boldsymbol{t}} = (0, d_1, d_2)$

$\left(R + \frac{\varepsilon}{2}tR\right) = \left[\left((c, s, 0{,}0) + \frac{\varepsilon}{2}(0)\right) + \frac{\varepsilon}{2}[(0, d_1, d_2)(c, s, 0{,}0)]\right]$

$= \left[(c, s, 0{,}0) + \frac{\varepsilon}{2}(0, cd_1 + sd_2, cd_2 - sd_1)\right] = \hat{q}_1$ (13)

And finally: $\hat{q}_v{}' = \left(R + \frac{\varepsilon}{2}tR\right)\hat{q}_v\overline{\left(R + \frac{\varepsilon}{2}tR\right)}^* = 1 + \frac{\varepsilon}{2}(0\,, 2d_1 + 2l\cos\theta\,, 2d_2 + 2l\sin\theta)$

$= 1 + \varepsilon\begin{pmatrix} 0 \\ d_1 + l\cos\theta \\ d_2 + l\sin\theta \end{pmatrix}$ And $\boldsymbol{v}' = \begin{pmatrix} 0 \\ d_1 + l\cos\theta \\ d_2 + l\sin\theta \end{pmatrix}$ $(\boldsymbol{b1})$

Using matrices (classical method):

$$[M]\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & d_1 \\ 0 & \sin\theta & \cos\theta & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ l \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ d_1 + l\cos\theta \\ d_2 + l\sin\theta \\ 1 \end{pmatrix} \Leftrightarrow (\mathrm{b}\mathbf{1})$$

2) The same translation $\vec{\boldsymbol{t}}$ followed by the same rotation R applied to the same vector $\hat{q}_v$;

$\left(R + \frac{\varepsilon}{2}Rt\right)\hat{q}_v\overline{\left(R + \frac{\varepsilon}{2}Rt\right)}^*$

$\left(R + \frac{\varepsilon}{2}Rt\right) = \left[(c, s, 0{,}0) + \frac{\varepsilon}{2}(0) + \frac{\varepsilon}{2}\left((c, s, 0{,}0) + \frac{\varepsilon}{2}(0)\right)\left(\frac{\varepsilon}{2}(0, d_1, d_2)\right)\right]$

$= \left[(c, s, 0{,}0) + \frac{\varepsilon}{2}(0, cd_1 - sd_2, cd_2 + sd_1)\right] = \hat{q}_2$ (14)

And finally the result is: $\left(R + \frac{\varepsilon}{2}Rt\right)\hat{q}_v\overline{\left(R + \frac{\varepsilon}{2}Rt\right)}^* =$

$= 1 + \frac{\varepsilon}{2}(0, d_1\cos\theta + l\cos\theta - d_2\sin\theta\,, d_1\sin\theta + l\sin\theta + d_2\cos\theta)$ $(\boldsymbol{b2})$

Checking the result using the classical method :

$$M\vec{\boldsymbol{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & \cos\theta\, d_1 - \sin\theta\, d_2 \\ 0 & \sin\theta & \cos\theta & \sin\theta\, d_1 + \cos\theta\, d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ l \\ 0 \\ 1 \end{pmatrix} = \left|\begin{matrix} 0 \\ d_1\cos\theta + l\cos\theta - d_2\sin\theta \\ d_1\sin\theta + l\sin\theta + d_2\cos\theta \\ 1 \end{matrix}\right. \Leftrightarrow (\boldsymbol{b2})$$

3) The same rotation R about a point C (a short cut for: about an axis getting through C and parallel to the axis of rotation) .See **Fig. (2)**

$$\hat{q} = \left(1 + \frac{\varepsilon}{2}\vec{\boldsymbol{d}}\right)R\left(1 - \frac{\varepsilon}{2}\vec{\boldsymbol{d}}\right) = R - \frac{\varepsilon}{2}R\vec{\boldsymbol{d}} + \frac{\varepsilon}{2}\vec{\boldsymbol{d}}R = R - \frac{\varepsilon}{2}\left(R\vec{\boldsymbol{d}} - \vec{\boldsymbol{d}}R\right);$$

$(-\vec{\boldsymbol{d}})$ is the displacement that takes the point C to the origin of the fixed frame and $(\vec{\boldsymbol{d}})$ its inverse, (which is equivalent to the conjugation TMT^{-1} technique). It has an inverse: $R^T + \frac{\varepsilon}{2}(R^T\boldsymbol{d} - \vec{\boldsymbol{d}}R^T)$

Same example as 1) and 2); C a point ϵ plane (y, z); $\vec{\boldsymbol{d}} = (0, y, z)^t$

Thus we have $\hat{q} = [(c, s, 0{,}0) + \varepsilon(0, 0, zs, -ys)] = \hat{q}_3 = \hat{q}_{d3}$ a rotation about $\overrightarrow{\boldsymbol{Cx}}$ axis $\parallel to\ \overrightarrow{\boldsymbol{ox}}$ (15)

$\hat{q}.v.\bar{\hat{q}}^* = [(c, s, 0{,}0) + \varepsilon(0{,}0, zs + cl, -ys + sl)][(c, -s, 0{,}0) + \varepsilon(0{,}0, zs, -ys)]$

$= 1 + \varepsilon(0, lcos\theta + zsin\theta + 2s^2y, lsin\theta - ysin\theta + 2s^2z)$

Knowing that $2s^2\left(\frac{\theta}{2}\right) = 1 - cos\theta$; thus the rotated vector $\vec{\boldsymbol{v}'}$ will be:

$$\vec{\boldsymbol{v}'} = \begin{pmatrix} 0 \\ lcos\theta + zsin\theta + y - ycos\,\theta \\ lsin\theta - ysin\theta + z - zcos\theta \end{pmatrix} \quad (\boldsymbol{b3})$$

This result could be checked using conjugation matrices TMT^{-1} : translate the point C to the origin O apply the rotation $M = R$ and retranslate it back to its initial position.

$$TRT^{-1}.\boldsymbol{v} = \begin{pmatrix} 0 \\ lcos\theta + zsin\theta + y - ycos\theta \\ lsin\theta - ysin\theta + z - zcos\theta \\ 1 \end{pmatrix} \Leftrightarrow (\boldsymbol{b3})$$

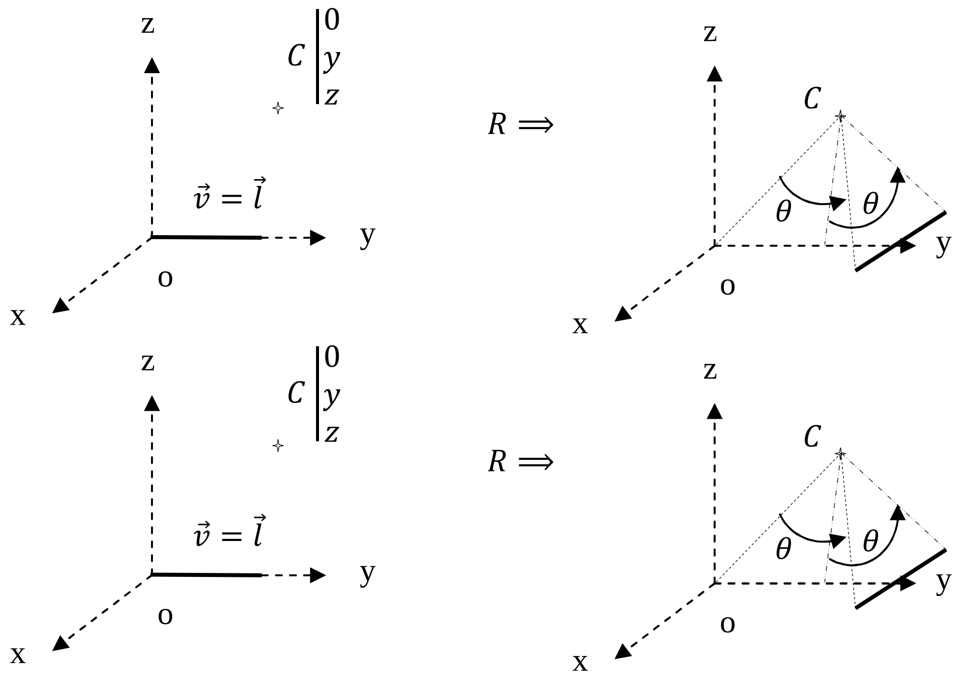


Fig (2)

We can observe that our three precedent examples could be all considered as planar displacements in the (y-z) plane shown in **Fig. (3)** Let d_y and d_z denote the coordinates of the origin O' of the moving frame M in the fixed frame F and θ denotes the rotation angle of M relative to F. Then these planar displacements can be represented by the following planar dual quaternions $\hat{q}$:

$\hat{q} = (q_1,q_2,q_3,q_4) + \frac{\varepsilon}{2}(q_5,q_6,q_7,q_8)$: where $q_1 = \cos(\theta/2)$; $q_2 = \sin(\theta/2)$; and $q_3 = q_4 = q_5 = q_6 = 0$: Such that for :

*Example 1) : $d_y = d_1$ and $d_z = d_2$; $q_7 = d_1\cos(\theta/2) + d_2\sin(\theta/2)$ *and* $q_8 = -d_1\sin(\theta/2) + d_2\cos(\theta/2)$

*Example 2): $d_y = d_1\cos(\theta) - d_2\sin(\theta)$ *and* $d_z = d_1\sin(\theta) + d_2\cos(\theta)$; $q_7 = (cd_1 - sd_2)$ and $q_8 = (cd_2 + sd_1)$

*Example 3) $d_y = y(1-cos(\theta)) + z\sin(\theta)$ and $d_z = z(1-cos(\theta)) - y\sin(\theta)$; $q_7 = 2zs$ *and* $q_8 = -2ys$

They can be considered as a set of homogeneous coordinates that define the image space of the three main planar displacements.

Note 1: As a matter of fact these three elementary planar transformations (1 , 2 and 3) could be applied to any vector $\overrightarrow{\boldsymbol{MN}} = (o,y,z)^t$ belonging to the plane yz.

Note 2 : The rotation R could be applied around the y-axis while $\overrightarrow{\boldsymbol{MN}} = (x,o,z)^t$ and center C belonging to the plane xz or R around the z-axis while $\overrightarrow{\boldsymbol{MN}} = (x,y,o)^t$ and center C belonging to the plane xy.

Note 3: The moving frame (***M)*** does represent any rigid body firmly attached to it.

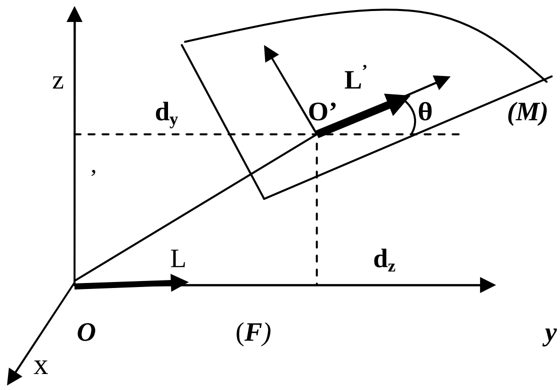


Fig. 3. Planar transformations

4) General Screw Motions $\hat{S}$ (without displacements or through a fixed center C) :
A screw motion is a rotation R around an axis **n** through the center C followed by a translation $\boldsymbol{t}$ (or a translation $\boldsymbol{t}$ followed by a rotation R around the same axis).Using either ((12 B) or (12 A); such that: the vector translation $\boldsymbol{t} = (t_x, t_y, t_z)$ is ∥ to the axis of the rotation **n** ; $(R = (c, s\mathbf{n}))$. We have using quaternions: $\hat{S} = R + \frac{\varepsilon}{2}R\boldsymbol{t} = R + \frac{\varepsilon}{2}\boldsymbol{t}R = \left[(c, s\mathbf{n}) + \frac{\varepsilon}{2}(-st, ct\mathbf{n})\right]$
Using dual quaternions DQ we have :

$$\hat{S} = \left[\left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right) + \varepsilon(0)\right]\left[1 + \frac{\varepsilon}{2}\left[(\mathrm{n}_x t, n_y t, n_z t)\right]\right] =$$

$$\left[\left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right) + \frac{\varepsilon}{2}(-st, ct\mathbf{n})\right] = \left[\left(\cos\frac{\theta}{2}, \mathbf{n}\sin\frac{\theta}{2}\right) + \frac{\varepsilon}{2}(-st, ct\mathbf{n})\right] \qquad (16)$$

We can notice that only one line is sufficient and contains explicitly all the informations while a (4x4) complicated matrix is needed to describe this general screw transformation.
To conclude; assume some oriented line $\overrightarrow{\boldsymbol{n}}$ in 3D space (as an axis of rotation passing through the point C, the solid (the cube in **Fig. (4)**) must undergo a screw motion):That is it must rotate an angle θ around the rotation axis and simultaneously translate the distance **t** (parallel to the same axis $\overrightarrow{\boldsymbol{n}}$).

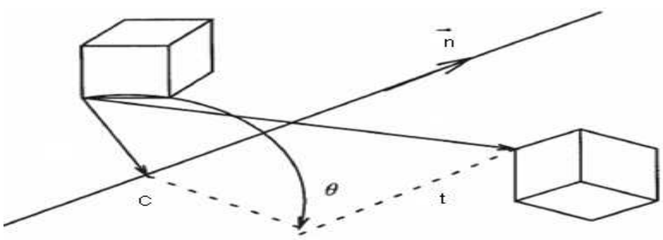


Fig. (4) General Screw Motion

This figure illustrates well the "Theorem of Chasles": *Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis"*.
Now the question to be asked is: Are applications r), 1) ,2) and 3) screw motions like 5) ?, and the answer is yes indeed:They are all rotations about some axis combined with translation **t =** zero.

4. CENTERS OF PLANAR RIGID TRANSFORMATIONS AND THEIR MOMENTS :

a) **Center of a transformation using matrices :**

A center of a transformation is the unique point C , left unchanged after the transformation [M] :
We will have for the example 1): $[M_1] = \overrightarrow{\boldsymbol{d}}\,[R]$
The coordinates of the center of transformation C_1 are defined by the well known equation:

$(R - I)\,\overrightarrow{\boldsymbol{C}} = -\,\overrightarrow{\boldsymbol{d}}$ Let $\vec{\boldsymbol{c}} = \left|\begin{matrix} X \\ Y \\ Z \end{matrix}\right.$; be the Cartesian coordinates of the center point C_1

Using Cramer's method or any other techniques the solution to this equation system is (for $\theta \neq 0$) : $\overrightarrow{oC_1} =$

$$\left| \begin{array}{l} X_1 = 0 \\ Y_1 = \frac{(1-\cos\theta)d_1 - \sin\theta d_2}{2(1-\cos\theta)} \\ Z_1 = \frac{\sin\theta d_1 + (1-\cos\theta)d_2}{2(1-\cos\theta)} \end{array} \right.$$

Using half angles we have : $\overrightarrow{oC_1} = \left| \begin{array}{l} X_1 = 0 \\ Y_1 = \frac{d_1}{2} - \frac{d_2}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} \\ Z_1 = \frac{d_1}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} + \frac{d_2}{2} \end{array} \right.$

Example 2) : Compared to example 1) : the same translation $\vec{d}$ followed by the same rotation R

$[M_2] = [R]\,\vec{d}$

Using the same technique to find the center of transformation , we will have ($\theta \neq 0$)

$$\overrightarrow{oC_2} = \left| \begin{array}{l} X_2 = 0 \\ Y_2 = \dfrac{(1-\cos\theta)d_3 - \sin\theta d_4}{2(1-\cos\theta)} = \dfrac{(\cos\theta - 1)d_1 - \sin\theta d_2}{2(1-\cos\theta)} \\ Z_2 = \dfrac{\sin\theta d_3 + (1-\cos\theta)d_4}{2(1-\cos\theta)} = \dfrac{\sin\theta d_1 + (\cos\theta - 1)d_2}{2(1-\cos\theta)} \end{array} \right.$$

Again using half angles we can have: $\overrightarrow{oC_2} = \left| \begin{array}{l} X_2 = 0 \\ -\frac{d_1}{2} - \frac{d_2}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} \\ \frac{d_1}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} - \frac{d_2}{2} \end{array} \right.$

a-bis) Centers of rigid transformations using dual quaternioms

Example 1):

$\hat{q}_1 = \left[(c, s, 0,0) + \frac{\varepsilon}{2}(0,\ cd_1 + sd_2, cd_2 - sd_1)\right]$

The center of transformation is the DQ vector left unchanged after the transformation :

$\hat{q}_c = 1 + \frac{\varepsilon}{2}(2x, 2y, 2z)$

then $\hat{q}_1 \,.\, \hat{q}_{c1} \,.\, \overline{\hat{q}_1} \equiv \hat{q}_{c1}$

$\hat{q}_1 \,.\, \hat{q}_{c1} \,.\, \overline{\hat{q}_1} = \left[(c, s, 0,0) + \frac{\varepsilon}{2}(0, cd_1 + sd_2, cd_2 - sd_1)\right]\left[1 + \frac{\varepsilon}{2}(2x, 2y, 2z)\right]\left[(c, -s, 0,0) + \frac{\varepsilon}{2}(0, cd_1 + sd_2, cd_2 - sd_1)\right] = \left[1 + \frac{\varepsilon}{2}(2x, 2y, 2z)\right]$

We will end up by a system of equations that could be solved using the usual techniques and the solution is :

$$\overrightarrow{oC_1} = \left| \begin{array}{l} X_1 = 0 \\ Y_1 = \dfrac{d_1}{2} - \dfrac{d_2}{2} \dfrac{\cos(\theta/2)}{\sin(\theta/2)} \\ Z_1 = \dfrac{d_1}{2} \dfrac{\cos(\theta/2)}{\sin(\theta/2)} + \dfrac{d_2}{2} \end{array} \right.$$

The same work can be done for the example 2) : $\overrightarrow{oC_2} = \left| \begin{array}{l} X_2 = 0 \\ -\frac{d_1}{2} - \frac{d_2}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} \\ \frac{d_1}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} - \frac{d_2}{2} \end{array} \right.$

b) <u>Moments of vectors</u>: Let us calculate the moment $\vec{m}$ of any vector $\overrightarrow{op}$, with the point P belonging to the C_x- axis , with respect to the O_x-axis : $\vec{m} = \overrightarrow{op} \wedge \vec{i} = \left(\overrightarrow{oC} + \overrightarrow{Cp}\right) \wedge \vec{i}$

Example 1): $\overrightarrow{m_1} = (\overrightarrow{oc_1} + \overrightarrow{c_1 p}) \wedge \vec{i} = \left| \begin{array}{l} X \\ \frac{d_1}{2} - \frac{d_2}{2}\frac{c}{s} \\ \frac{d_1}{2}\frac{c}{s} + \frac{d_2}{2} \end{array} \right. \wedge \left| \begin{array}{l} 1 \\ 0 \\ 0 \end{array} \right. = \left| \begin{array}{c} 0 \\ \frac{d_1}{2}\frac{c}{s} + \frac{d_2}{2} \\ -\frac{d_1}{2} + \frac{d_2}{2}\frac{c}{s} \end{array} \right.$

Example 2): $\overrightarrow{m_2} = (\overrightarrow{oc_2} + \overrightarrow{c_2 p}) \wedge \vec{i} = \left| \begin{array}{c} x \\ -\frac{d_1}{2} - \frac{d_2}{2}\frac{c}{s} \\ \frac{d_1}{2}\frac{c}{s} - \frac{d_2}{2} \end{array} \right. \wedge \left| \begin{array}{l} 1 \\ 0 \\ 0 \end{array} \right. = \left| \begin{array}{c} 0 \\ \frac{d_1}{2}\frac{c}{s} - \frac{d_2}{2} \\ \frac{d_1}{2} + \frac{d_2}{2}\frac{c}{s} \end{array} \right.$

Multipluing these two vectors by the quantity $\sin{}^{\theta}/_{2}$ will give us exactly the dual parts of the dual planar quaternions $\hat{q}_1$ and $\hat{q}_2$ respectively :

example 1): $\overrightarrow{\boldsymbol{m_1}} \sin(\theta/2) = 1/2 \left| \begin{matrix} 0 \\ d_1 c + d_2 s \\ -d_1 s + d_2 c \end{matrix} \right.$

example 2): $\overrightarrow{\boldsymbol{m_2}} \sin(\theta/2) = 1/2 \left| \begin{matrix} 0 \\ d_1 c - d_2 s \\ d_1 s + d_2 c \end{matrix} \right.$

example 3):

The most important thing to notice at this stage is that if we are given a planar dual quaternion, like $\hat{q}_1$ or $\hat{q}_2$, we can instantly notice its orientation axis $\vec{\boldsymbol{n}}$ and its amplitude from the rotation quaternions q_{rot} and then considering the dual part of the planar DQ and dividing it by $\sin \theta/2$, we will have the vector moment :

$\overrightarrow{\boldsymbol{m}} = \overrightarrow{\boldsymbol{op}} \wedge \vec{\boldsymbol{n}} = \frac{1}{\sin\theta/2}\left(x\vec{\boldsymbol{\imath}}, y\vec{\boldsymbol{\jmath}}, z\vec{\boldsymbol{k}}\right)$

We can then get easily the coordinates of the fixed centers C of any planar transformations) by solving the equation:

$\overrightarrow{\boldsymbol{op}} = (\overrightarrow{\boldsymbol{oc}} + \overrightarrow{\boldsymbol{cp}}) = \frac{-(\vec{m} \wedge \vec{n})}{n^2}$.

So for our examples 1) and 2) the planar quaternions describing the motions are all of the form :

$$\hat{q} = (c, s, 0, 0) + \frac{\varepsilon}{2}\left(\overrightarrow{\boldsymbol{m}} \sin \theta/2\right) \quad (17)$$

In the case of a planar dual quaternion, we can rapidly read all the necessary informations about the transformation: A rotation of amplitude θ about an axis $\vec{\boldsymbol{n}}$ keeping its orientation and passing through a fixed center C defined by $\overrightarrow{\boldsymbol{oc}} = \frac{-(\vec{m} \wedge \vec{n})}{n^2}$.

Thus applying both displacements components a translation $\vec{\boldsymbol{t}} \parallel \mathbf{n}$ and a planar displacement $\vec{\boldsymbol{d}}$: $(\vec{\boldsymbol{t}} + \vec{\boldsymbol{d}})$ simultaneously; the dual quaternion representing this screw motion is simply :

$$\hat{q} = \hat{s} = (c, s\boldsymbol{n}) + \frac{\varepsilon}{2}(\boldsymbol{t} + \boldsymbol{d})(c, s\boldsymbol{n}) =$$
$$(c, s\boldsymbol{n}) + \left[\frac{\varepsilon}{2}(\boldsymbol{t})(c, s\boldsymbol{n}) + \frac{\varepsilon}{2}(\boldsymbol{d})(c, s\boldsymbol{n})\right] = (c, s\boldsymbol{n}) + \left[\frac{\varepsilon}{2}(-st, ct\boldsymbol{n}) + \frac{\varepsilon}{2}(\overrightarrow{\boldsymbol{m}}s)\right] =$$
$$(c, s\boldsymbol{n}) + \frac{\varepsilon}{2}(-st, \overrightarrow{\boldsymbol{m}}s, ct\boldsymbol{n}) \quad (18)$$

This is the general representation of a screw motion :(its counterpart in matrices;see AppendixV) : A rotation $R = (c, s\boldsymbol{n})$ followed by a translation $\boldsymbol{t}$ along the **n**-axis being displaced to a new position $\overrightarrow{OC}$ defined by its moment $\overrightarrow{\boldsymbol{m}}$ w.r.t to the origin O *and* C being enabled to translate the quantity **t** along the axis of rotation with a given amplitude θ (or a given pitch).The pitch of this screw motion is $p = \frac{2\pi}{\theta} t$

<u>Some remarks :</u>See **Fig. (5)**

$\hat{S} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{\boldsymbol{n}}\right) + \frac{\varepsilon}{2}\left(-t\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\overrightarrow{\boldsymbol{m}} + t\cos\frac{\theta}{2}\vec{\boldsymbol{n}}\right)$

- If $\theta = 0$: $\hat{q} = 1 + \frac{\varepsilon}{2}\boldsymbol{t}$ A pure translation **t**
- $\mathbf{t} = 0$, $\mathbf{m} = 0$: $\hat{q} = (c, s\boldsymbol{n})$ A pure rotation of angle θ around the **n** -axis through the origin O , $p = 0$
- $\boldsymbol{t} = 0, \mathbf{m}$: A pure rotation of angle θ around an axis parallel to **n** through the center C . $p = 0$
- $\boldsymbol{m} = 0$: A pure screw motion about the axis **n** *through O* with pitch $p = \frac{2\pi}{\theta}\boldsymbol{t}.\mathbf{n} = \frac{2\pi}{\theta} t$
- $\boldsymbol{t}, \mathbf{m}$: A general screw motion about the axis **n** *through a fixed point C* with pitch $p = \frac{2\pi}{\theta}\boldsymbol{t}.\mathbf{n}$.
-

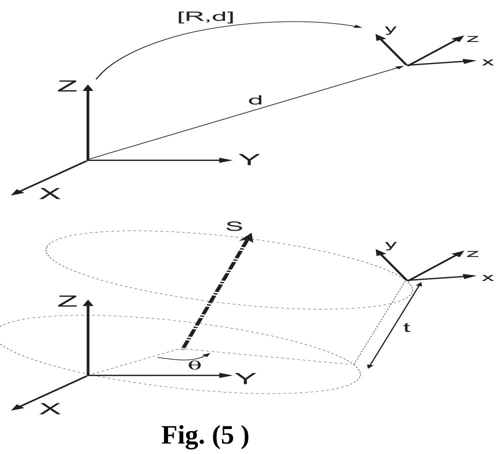


Fig. (5)

5. KINEMATIC EQUATION OF SERIAL ROBOTS

A serial robot is a sequence of rigid links connected by joints to form a serial open chain. One end is fixed to the base. The other end is equipped with an end-effector. The joints are generally powered and designed to control the position of a link relative to its neighbour. The number of actuators needed to define the configuration of the chain is called its degree-of-freedom (DOF).

For serial chains, DOF is equal to the number of joints in the chains, counting the DOF of compound joints.

A) Forward kinematic model in dual quaternion space

To derive the forward kinematic model (FKM) of a serial link manipulator, the standard Denavit-Hartenberg convention that uses four parameters has widely been used.

These parameters a_{i-1}, α_{i-1}, d_i and θi are the link length, link twist, link offset and joint angle, respectively. A coordinate frame is attached to each joint to determine DH parameters.

Zi axis of the coordinate frame is pointing along the rotary or sliding direction of the joints. Having assigned coordinate frames to each link, the transformation between two successive frames F_{i-1} and F_i is described with the following rotations and translations (Craig, 1989):

1) Translation along axis z_{i-1} of a distance d_i: $Trans(z_{i-1}, d_i)$
2) Rotation about axis z_{i-1} of an angle θ_i : $Rot(z_{i-1}, \theta_i)$
3) Translation along axis x_{i-1} of a distance a_{i-1}: $Trans(x_{i-1}, a_{i-1})$
4) Rotation about axis x_{i-1} of an angle α_{i-1}: $Rot(x_{i-1}, \alpha_{i-1})$

Similarly, in the dual quaternion space, the transformation between these frames is obtained by multiplication of the 4 dual quaternions corresponding to each transformation:

$$q_{i-1,i} = q_{rot}(x_{i-1}, \alpha_{i-1}) q_{trans}(x_{i-1}, a_{i-1}) q_{rot}(z_{i-1}, \theta_i) q_{trans}(z_{i-1}, d_i)$$

The subscripts “rot” and “trans” indicate whether the transformation is a pure rotation or a pure translation, respectively.In these cases, the dual quaternions are simplified. For a pure rotation, the translation vector is 0 and thus $q_{rot} = q_{rot}$. For a pure translation, the rotation quaternion is the identity and hence $q_{trans} = 1 + \varepsilon \frac{t}{2}$

The most important thing to notice at this stage is that this multiplication corresponds to the multiplication of ***two screw motions***: the first one around z_{n-1} axis with angle (θ_n) and translation $\mathbf{t_1} = \boldsymbol{d_i}$ and the second one around x_{i-1} axis with angle (α_{i-1}) and translation $\mathbf{t_2} = \boldsymbol{a_{i-1}}$

The FKM can be then calculated for an n-link robot as:

$$q = q_{01}\, q_{12} \cdots q_{n-1,n}$$

The inverse kinematic model (IKM) could be performed to evaluate joints variables.

B) Jacobian matrix in the dual quaternion space

For any robot manipulator with n joints, the kinematic expression that relates the end-effector velocity vector X' to the joint velocity vector $\boldsymbol{\theta}'$ is given by the well-known relationship:

$$X' = \boldsymbol{J}(\theta)\, \boldsymbol{\theta}'$$

Where the Jacobian matrix $\boldsymbol{J}(\theta)$ is given analytically by $\boldsymbol{J}(\theta) = \frac{dX}{d\theta}$

In the dual quaternion space the relationship becomes: $\widehat{\boldsymbol{q}} = J(q)\boldsymbol{\theta}'$

Now the Jacobian matrix relating the joint velocity vector $\boldsymbol{\theta}'$ and the time derivative of the dual quaternion $\widehat{\boldsymbol{q}}$ is given analytically by

$$J(q) = \frac{d\widehat{\boldsymbol{q}}}{d\boldsymbol{\theta}} = \frac{d}{d\boldsymbol{\theta}} \begin{vmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{vmatrix} = \left(\begin{pmatrix} \frac{dq_1}{d\theta_1} & \cdots & \frac{dq_1}{d\theta_n} \\ \vdots & \ddots & \vdots \\ \frac{dq_8}{d\theta_1} & \cdots & \frac{dq_8}{d\theta_n} \end{pmatrix} \right) ;\ (8 \times n)\ matrix$$

Where $\boldsymbol{q} = [q_1\ q_2\ q_3\ q_4]^T$ and $\boldsymbol{q_0} = [q_5\ q_6\ q_7\ q_8]^T$, with $\boldsymbol{q}$ and $\boldsymbol{q_0}$ being the components of the dual quaternion $\widehat{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon \boldsymbol{q_0}$.

6. Applications:

1) Consider a **planar manipulator *RRR*** in its home position: (see Appendix III): DQ-Matlab-Computation)

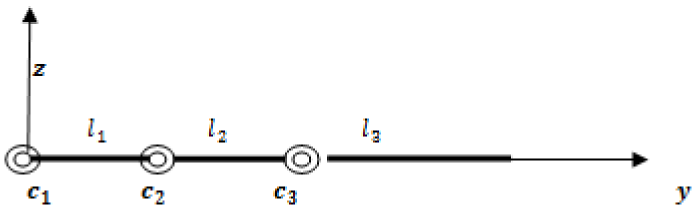


Fig. 6 Home position

***Link l_3* :** This last link rotates around the c_{x3}-axis an angle θ_3, the center of rotation has the coordinates: $(0, l_1 + l_2, 0)^T$ (**Fig.** . (**6**)) , while the end effector (wrist) is at $(0, l_1 + l_2 + l_3, 0)^T$
Using (see Example III-3), the dual quaternion $\hat{q}_{\,d3}$:
$\hat{q}_{\,d3} = [(\,c, s, 0{,}0\,) + \varepsilon\,(\,0, zs, -ys)]$ = R + $\frac{\varepsilon}{2}(0, 2\mathbf{m}\; s\,)$ = R+ ε (0, **m** s) with **m** being the moment of **OC₃** = (0, y , z)^T w.r.t to the axis Ox .With x = z = 0 and y = $l_1 + l_2$,we have; $\hat{q}_{\,d3}\,.\,\widehat{v_3} = [(c_3, s_3, 0{,}0) + \varepsilon(\,0{,}0, -(l_1 + l_2)s_3)][1 + \varepsilon\,(0, (l_1 + l_2 + l_3), 0)] =$
$[(c_3, s_3, 0{,}0) + \varepsilon(0, (l_1 + l_2 + l_3)c_3, l_3 s_3)]$ and then multiplying by the conjugate $\overline{\widehat{q_{d3}}^*}$ we will end up by : (using all along these applications the usual trigonometric properties and rules)
$\widehat{v_2} = \hat{q}_{\,d3}\,\widehat{v_3}\,\overline{\widehat{q_{d3}}^*} = 1 + \varepsilon(\,0, l_1 + l_2 + cos\theta_3 l_3\,, l_3 sin\theta_3)$
Link l_2 : $\hat{q}_{\,d2}$: Rotation of θ_2 arround the axis c_{x2}; $\overrightarrow{oc_2} = (0, l_1, 0)^T$
Using the same technique : $\hat{q}_{\,d2}\,\widehat{v_2}\,\overline{\widehat{q_{d2}}^*}$ we will have the result :
$\widehat{v_1} = 1 + \varepsilon\,(\,0, l_1 + l_2 cos\theta_2 + \cos(\theta_2 + \theta_3)\,l_3\,, \sin(\theta_2 + \theta_3)\,l_3 + \sin\theta_2 l_2)$
Link l_1 : And finally the last link l_1 must rotate around the c_{x1} or o_x-axis ; the amplitude being θ_1. S_0 the transformation is : $\widehat{v_0} = \hat{q}_{\,d1}\,\widehat{v_1}\,\overline{\widehat{q_{d1}}^*} = [(c_1, s_1, 0{,}0) + \varepsilon(0{,}0{,}0)]\,\widehat{v_1}\,[(c_1, -s_1, 0{,}0) + \varepsilon(0{,}0{,}0)] = 1 + \varepsilon(0, cos\theta_1 l_1 + \cos(\theta_1 + \theta_2)\,l_2 + \cos(\theta_1 + \theta_2 + \theta_3)\,l_3\,, sin\theta_1 l_1 + \sin(\theta_1 + \theta_2)\,l_2 + \sin(\theta_1 + \theta_2 + \theta_3)l_3)$

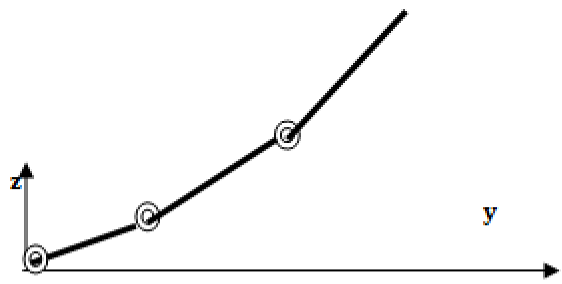


Fig. 7 Using (4x4) matrices and conjugation techniques

Or directly: $\widehat{v_0} = [\hat{q}_{\,d1}.\hat{q}_{\,d2}.\hat{q}_{\,d3}\,].\,\widehat{v_3}\,.\,[\overline{\widehat{q_{d3}}^*}\,.\,\overline{\widehat{q_{d2}}^*}\,.\,\overline{\widehat{q_{d1}}^*}]$
We can check these results (**Fig. (7)**) using (4x4) matrices and conjugation techniques.

2) **The 3R Wrist**.
This structure is used as a wrist in several robots, PUMA, STANFORD, SCARA and many others. The three joint axes intersect at a common point C, the mechanism is thus a spherical one. . The arrangement (home position) of the joints is illustrated in **Fig. (8)**.

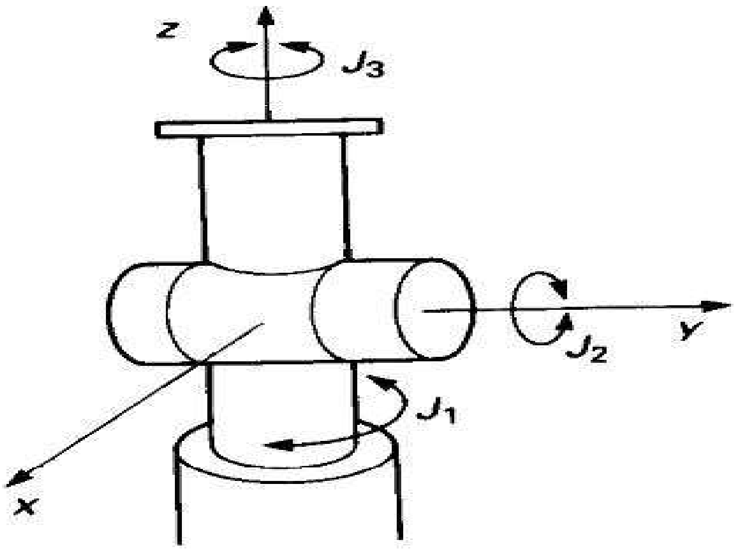


Fig. (8): ***The (ZYZ) Wrist***

Choosing the Euler angles (YZY) sequence instead of the (ZYZ) of the figure above; the kinematics of the wrist will be:

i) Using quaternionss: $\hat{q}_v{}'$ = ($q_1.q_2.q_3$) . $\hat{q}_v$. ($q_3^*.q_2^*.q_1^*$) where:
q_3=(c_3,0,0,s_3),q_2=(c_2,0,s_2,0) ,q_1=(c_1,0,0,s_1) and q_3^*=(c_3,0,0,$-s_3$), q_2^*=(c_2,0,$-s_2$,0) , q_1^*= (c_1,0,0, $-$ s_1)
ii) Using matrices $K(\theta_1,\theta_2,\theta_3) = R(\theta_1,\boldsymbol{k})R(\theta_2,\boldsymbol{j})R(\theta_3,\boldsymbol{k})$
Computing the quaternion product ($q_1.q_2.q_3$) and converting it to a matrix will give the same (See appendix IV) result, which corresponds to the famous “Euler angles representation “ with its singularities and “gimbal lock”.

3**) 6R robot**.

Using the precedent results a 6R manipulator could be designed by adjusting the above wrist center C = C_4 at a distance $(0, l_1 + l_2 + l_3, 0)^T$ (the End Effector of the RRR planar manipulator).

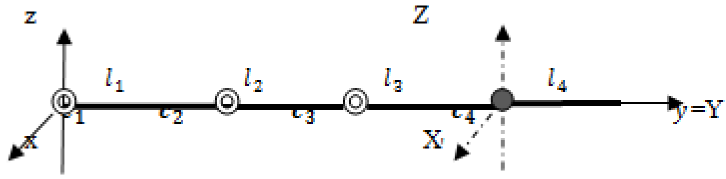


Fig. 9. The vector position of some tool rigidly attached to the gripper has the home position

The forward kinematics model of this 6R robot (FKM) is: $\hat{q}_{d1}.\hat{q}_{d2}.\widehat{q}_{d3}.\widehat{q}_{d4}.\hat{q}_{d5}.\widehat{q}_{d6}$
The vector position of some tool rigidly attached to the gripper has the home position (**Fig. (9)**):
$(0, l_1 + l_2 + l_3 + l_4, 0)^T = (0, L + l_4, 0)^T$
Its first transformation is only a pure rotation around the Y_6-axis ;it can simply be represented by the quaternion q_6 =(c_6,0,s_6,0), it does also rotate around the Z_5-axis parallel to z-axis and this is represented by the dual quaternion $\hat{q}_{d5}$= (c_5,0, 0 , s_5) + ε(0, **m** s_5) ; **m** being the moment of **OC_4 =** (x, y , z)^T = (0, L ,0)^T w.r.t to the axis Oz; $\hat{q}_{d5}$= (c_5,0,0,s_5) + ε(Ls_5,0, 0). The third rotation around the Y_4-axis can be represented by the quaternion q_4 =(c_4,0,s_4,0). The first operation will give: $\widehat{v_5} = q_6.\widehat{v_6}.q_6^* = 1 + \varepsilon(0, L + l_4, 0) = \widehat{v_6}$; an unchanged vector.
The second transformation is: $\widehat{v_4} = \hat{q}_{d5}.\widehat{v_5}.\overline{\widehat{q_{d5}}^*} = 1 + \varepsilon(-l_4 sin\theta_5,\ L + l_4 cos\theta_5\ ,0)$ and finally the third wrist rotation will result in: $\widehat{v_3} = q_4.\widehat{v_4}.q_4^* = 1 + \varepsilon(-l_4 cos\theta_4 sin\theta_5,\ L + l_4 cos\theta_5,\ l_4 sin\theta_4 sin\theta_5)$ then **v_3** = (x_3, y_3, z_3)^T.
The robot fourth motion (back to our planar manipulator) will be around c_{x3}-axis with angle θ_3, the center of rotation has the coordinates: $(0, l_1 + l_2, 0)^T$ .Using the trigonometric identities: We have; $\widehat{v_2} =$

$\hat{q}_{d3}\,.\widehat{v_3}.\overline{\widehat{q_{d3}}^*} = [(c_3, s_3, 0,0) + \varepsilon(0,0,-(l_1+l_2)s_3)][+\varepsilon((x_3, y_3, z_3))]\widehat{q_{d3}}^* = 1 + \varepsilon(x_3, (1 - cos\theta_3)(l_1+l_2) + cos\theta_3 y_3 - sin\theta_3 z_3, -sin\theta_3(l_1+l_2) + sin\theta_3 y_3 + cos\theta_3 z_3)$ and replacing x_3, y_3 and z_3 by their respective values we can get the transformed vector:
$\mathbf{v_2} = (-l_4 cos\theta_4 sin\theta_5, l_1 + l_2 + l_3 cos\theta_3 + l_4 cos\theta_3\, cos\theta_5 - l_4 sin\theta_3 sin\theta_4 sin\theta_5,$
$l_3 sin\theta_3 + l_4\, sin\theta_3 cos\theta_5 + l_4\, cos\theta_3\, sin\theta_4 sin\theta_5)^T = (x_2, y_2, z_2)^T$.
The fifth rotation is around the c_{x2}-axis with angle θ_2 and center at: $(0, l_1, 0)^T$.
Following the same technique the DQ representing this transformation is: $\widehat{v_1} = \hat{q}_{d2}\,.\widehat{v_2}.\overline{\widehat{q_{d2}}^*} =$
$[(c_2, s_2, 0,0) + \varepsilon(0,0,-(l_1)s_2)][1 + \varepsilon((x_2, y_2, z_2))]\widehat{q_{d2}}^* = 1 + \varepsilon(0, x_2, 2s_2^2 l_1 + (c_2^2 - s_2^2)y_2 - 2s_2c_2z_2, -2s_2c_2l_1 + 2s_2c_2y_2 + (c_2^2 - s_2^2)z_2)$
Replacing x_2, y_2, and z_2 by their values we then have the transformed vector $\mathbf{v_1} = (x_1, y_1, z_1)^T$.
And finally the last transformation is about the c_{x1}-axis or the Ox- axis with amplitude θ_1; thus it can be described by the quaternion q_1; $\widehat{v_0} = q_1.\widehat{v_1}.q_1^*$
We can also check these results using (4x4) matrices and conjugation techniques.

4) A better performance (Puma-like) 6R robot:
Allowing a rotation θ_1 around the Oz- axis instead of Ox- axis to get a better mobility(mobility) and an offset of distance d between the two parallel axes Y and y (**Fig. (10)**) our robot will be similar to the six axes industrial robot; the Unimate Puma. Thus the FKM of this robot type Puma is:
$\hat{Q} = \hat{q}_{d1}.\hat{q}_{d2}.\hat{q}_{d3}.\hat{q}_{d4}.\hat{q}_{d5}.\hat{q}_{d6}$ *with* $\hat{q}_{d1} = q_1 = (c_1 0,0, s_1)$, $\hat{q}_{d6} = (C_6, 0, S_6, 0) + \varepsilon(0, \mathbf{m}\, S_6) = (C_6, 0, S_6, 0) + \varepsilon(0,0, dS_6)$, $\hat{q}_{d5} = (C_5, 0, 0, S_5) + \varepsilon(0, \mathbf{m}\, S_5) = (C_5, 0, 0, S_5) + \varepsilon(LS_5, 0, -dS_5)$ and $\hat{q}_{d4} = (C_4, 0, S_4, 0) + \varepsilon(0,0, dS_4)$.We can check that the wrist center C_4 is unaffected (left unchanged) by these last three DQ transformations. Now we are in a position to express the forward and the inverse kinematics in terms of three points rigidly attached to the gripper with home frame coordinates: the shifted wrist center $\mathbf{OC_4} = (d, L, 0)^T$, $\mathbf{OA} = (d, L, a)^T$ and $\mathbf{OB} = (d + b, L, 0)^T$.

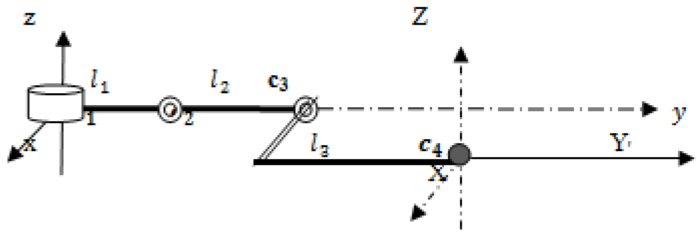


Fig. 10. Offset of distance d between the two parallel axes Y and y

Using the existing abundant literature about the (4x4) forward kinematic matrices used for this type of Unimate Puma robot, we can compare the two methods and check these results using conversion techniques (see Matlab Appendices).
This dual quaternion analysis can be generalized to any other type of serial robot provided its exact spatial ossature (architecture) is given (link lengths,rotations or/ and translations axes,rotation centers,offsets...) or in two words knowing its D-H parameters.

7. CONCLUSION

The kinematics equation of a serial chain defines the location of the end-effector in terms of the joint parameters.The position and orientation can be parameterized in different ways, such as with the well-known (4 × 4) homogeneous transformation matrix, Euler angles, unit quaternion or dual quaternion. In a homogeneous transformation matrix, twelve parameters are used to represent the position and orientation of a body. When the unit dual quaternion is adopted for the transformation, an eight-dimensional vector must be defined. Moreover, all types of Euler angles have a "rotation-insequence" nature. Thus, the Euler angle method is suitable for representing a single frame orientation but not for representing orientation paths in the case of trajectory tracking. Another critical issue for the Euler angles is that they suffer from representation singularities. The unit quaternion represents the end-effector orientation without singularities.

In the homogeneous transformation method, four trigonometric function calls and six multiplications are required for calculation of the transformation matrix t-1 $[T]_t$ t. The operation used in this algorithm is the product of (4 X 4) transformation matrices. The multiplication of two (4 X 4) transformation matrices needs 48 multiplications and 36 additions and subtraction, since the elements of the last row of the matrix are constants. In n-link robot arm the number of transformation matrices is n, so n-1 number of matrix products is required in order to detrmine the total transformation matrix. Hence, the determination of the end effector position and orientation needs (48(n-1) + 6n) multiplications and 36(n-1) additions and subtractions, while only the orientation needs 31(n-1) multiplications and 18(n-1) additions and subtractions.

For the case of 3-R robot (n=3), 114 multiplications and 72 additions and subtractions are necessary. This speaks about the number of mathematical operations required for computing the homogeneous matrix and hence the time needed for the same.

Despite the eight-dimension of the dual quaternion, many authors stated that the dual quaternion is the most compact and efficient way to express the screw motion, that is, both translational and rotational transformations in a robot kinematic chain. The dual quaternion turns out to be an elegant and useful tool for kinematic analysis in many researches such as in inertial navigation, graphics and computer vision.

Table 1. Performance comparison of rotation operations.

Method	Storage	Multiplies	Add & Substracts	Total
Rotation Matrix (3x3)	9	27	18	45
Quaternion	4	16	12	28

Table 2. Performance comparison of rigid transformation operations.

Method	Storage	Multiplies	Add & Substracts	Total
Homogeneous (4x4) Transformation Matrix	16	60	36	96
Dual Quaternion	8	48	40	84

It is evident from the results (**Table 1. and Table 2)** that a matrix product requires many more operations than a quaternion product. So a lot of time can be saved and at the same time more numerical accuracy can be preserved with quaternion than with matrices. In the examples mentioned in the present work, it is clear that quaternion algebra provides a very effective and efficient method for representation of forward kinematics equation. Further, the method is cost effective as it requires less computer memory and saves lot of time by reducing the number of mathematical calculation. Comparing the two methods, it is observed that the quaternion method gives exactly the same result as that of homogeneous method. This is a general method applied specifically to robot manipulators in the present work. However this can also be extended to any other open kinematic chain for the purpose of kinematic analysis. Therefore this can be used as a powerful tool in the solution of kinematic problems.

Note:In Matlab softwares we can find a gigantic library of applications dealing with all kind of matrices but only a very small one dealing with only quaternions (no DQs) and only with their numerical applications. Thus a lot of work has to be done before the birth and creation of an emergency powerfull “DualQuaternionlab” software.Presently there are many quaternion applications in the area of aerospace sequence, spherical trigonometry, calculus for kinematics and dynamics, rotation in phase space etc.A lot more research has to be done in this aspect and the days are not far behind when dual quaternions will replace the traditional homogeneous method of representation and mainly in the field of robotics.

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APPENDICES::

APPENDIX I: Conversions: Given a (4×4) homogeneous matrix $[T] = [R, \boldsymbol{d}]$, where $[R]$ and $\boldsymbol{d}$ are the rotation and translation part respectively, we can construct the corresponding dual quaternion as follows. First compute the rotation axis $\boldsymbol{S} = (S_x, S_y, S_z)^T$ and angle θ from the rotation matrix $[R]$ using Cayley's formula (McCarthy 1990). The real part quaternion q can then be constructed as:
$q = \left(\cos\frac{\theta}{2}, S_x \sin\frac{\theta}{2}, S_y \sin\frac{\theta}{2}, S_z \sin\frac{\theta}{2}\right)$

The image part quaternion q_0 is constructed as: $= \begin{pmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & q_4 & -q_3 \\ -q_4 & q_1 & q_2 \\ q_3 & -q_2 & q_1 \end{pmatrix} \boldsymbol{d}$

Such that:

$$q_5 = \left(-S_x \sin\frac{\theta}{2} dx - S_y \sin\frac{\theta}{2} dy - S_z \sin\frac{\theta}{2} dz\right)$$
$$q_6 = \left(\cos\frac{\theta}{2} dx + S_z \sin\frac{\theta}{2} dy - S_y \sin\frac{\theta}{2} dz\right)$$
$$q_7 = \left(-S_z \sin\frac{\theta}{2} dx + \cos\frac{\theta}{2} dy + S_x \sin\frac{\theta}{2} dz\right)$$
$$q_8 = \left(S_y \sin\frac{\theta}{2} dx - s_x \sin\frac{\theta}{2} dy + \cos\frac{\theta}{2} dz\right)$$

On the other hand, if we are given a dual quaternion $\hat{\boldsymbol{q}}$ subjected to the two well known constraints, the 4×4 homogenous transformation matrix $[T(\boldsymbol{q})] = [R(\boldsymbol{q}), d(\boldsymbol{q})]$ can be written as:

$$[R(\hat{\boldsymbol{q}})] = \begin{bmatrix} q_2{}^2 - q_3{}^2 - q_4{}^2 + q_1{}^2 & 2(q_2 q_3 - q_4 q_1) & 2(q_2 q_4 + q_3 q_1) \\ 2(q_2 q_3 + q_4 q_1) & -q_2{}^2 + q_3{}^2 - q_4{}^2 + q_1{}^2 & 2(q_3 q_4 - q_2 q_1) \\ 2(q_2 q_4 - q_3 q_1) & 2(q_3 q_4 + q_2 q_1) & -q_2{}^2 - q_3{}^2 + q_4{}^2 + q_1{}^2 \end{bmatrix}$$

And

$$d(\hat{\boldsymbol{q}}) = \begin{bmatrix} -q_5 & q_8 & -q_7 & q_6 \\ -q_8 & -q_5 & q_6 & q_7 \\ q_7 & -q_6 & -q_5 & q_8 \end{bmatrix} \begin{vmatrix} q_2 \\ q_3 \\ q_4 \\ q_1 \end{vmatrix}$$

APPENDIX II: *Matlab « Dual Quaternion » multiplication program*

```
syms q1
syms I
syms q2
syms q3
syms q4
syms qa
syms qb
qa=q1+I*q2
qb=q3+I*q4
Q=qa*qb
Q=q1*q3+I*(q2*q3+q1*q4)
```

APPENDIX III: 3R planar manipulator with matlab :

```
dq:dual quaternion
cq:conjugate quaternion

link(3)
syms c3
syms s3
syms l1
syms l2
syms l3
syms I
dq3=(c3,s3,0,0)+I*(0,0,-(l1+l2)*s3)
cq3=(c3,-s3,0,0)+I*(0,0,-(l1+l2)*s3)
v3=1+I*(0,(l1+l2+l3),0)
dq3*v3*cq3=1+I*(0,l1+l2+c3*l3;l3*s3)

link(2)
syms c2
syms s2
syms c12
syms s12
dq2=(c2,s2,0,0)+I*(0,0,-l1*s2)
cq2=(c2,-s2,0,0)+I*(0,0,-l1*s2)
v2=1+I*(0,l1+l2+c3*l3,l3*s3)
dq2*v2*cq2=1+I*(0,l1+l2*c2+c12*l3,s12*l3+s2*l2)

link(1)
syms c1
syms s1
syms c23
syms s23
syms c123
syms s123
dq1=(c1,s1,0,0)+I*(0,0,0)
cq1=(c1,-s1,0,0)+I*(0,0,0)
v1=1+I*(0,l1+l2*c2+l3*c23;l2*s2+l3*s23)
dq1*v1*cq1=1+I*(0,c1*l1+c12*l2+c123*l3,s1*l1+s12*l2+s123*l3)

..........................................................................................................
dq1=(c1,s1,0,0)+I*(0,0,0)
cq1=(c1,-s1,0,0)+I*(0,0,0)
dq2=(c2,s2,0,0)+I*(0,0,-l1*s2)
cq2=(c2,-s2,0,0)+I*(0,0,-l1*s2)
dq3=(c3,s3,0,0)+I*(0,0,-(l1+l2)*s3)
cq3=(c3,-s3,0,0)+I*(0,0,-(l1+l2)*s3)

a=(c1,s1,0,0)
b=(0,0,0)
c=(c2,s2,0,0)
d=(0,0,-l1*s2)

e=(c3,s3,0,0)
f=(0,0,-(l1+l2)*s3)
v3=1+I*(0,(l1+l2+l3),0)
```

APPENDIX IV: The YZY Wrist

```
syms t1
syms t2
syms t3
a1=[cos(t1) 0 sin(t1);0 1 0;-sin(t1) 0 cos(t1)]

a2=[cos(t2) -sin(t2) 0;sin(t2) cos(t2) 0;0 0 1]

a3=[cos(t3) 0 sin(t3);0 1 0;-sin(t3) 0 cos(t3)]
```

R=a1*a2*a3
R =
[cos(t1)*cos(t2)*cos(t3) - sin(t1)*sin(t3), -cos(t3)*sin(t2), cos(t1)*sin(t3) + cos(t2)*cos(t3)*sin(t1)]
[cos(t1)*sin(t2), cos(t2), sin(t1)*sin(t2)]
[- cos(t3)*sin(t1) - cos(t1)*cos(t2)*sin(t3), sin(t2)*sin(t3), cos(t1)*cos(t3) - cos(t2)*sin(t1)*sin(t3)]

Appendix V: A general Screw Motion.

$$\begin{pmatrix} n_x^2 v + c & n_x n_y v - n_z s & n_x n_z v + n_y s & t_x \\ n_x n_y v + n_z s & n_y^2 v + c & n_y n_z v - n_x s & t_y \\ n_x n_z v - n_y s & n_y n_z v + n_x s & n_z^2 v + c & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With : $v = 1 - cos\theta$, $c = cos\theta$, $s = sin\theta$